An Extension of Green's Theorem.

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§ 1. *Rectifiable Green Fields.*

In the usual proof of Green's theorem the functions must be continuous, and have at each point in the field of integration partial derivatives of the first order which are integrable over the given field. The field itself is assumed to have as a boundary a rectifiable curve, which is cut only a limited number of times by lines parallel to the axes. Other proofs* of this theorem have been given, in which the conditions on the functions are not so narrow, but all require the field of integration to satisfy the condition mentioned above. The proof of Green's theorem given in this paper makes the theorem apply to a much larger class of functions than has been possible before, and also permits the field of integration to be cut an infinite number of times by each one of a certain set of parallels.

For the sake of clearness, the definition of a line integral over any rectifiable curve will be given, together with some other geometric definitions.

Let $x = X(t)$, $y = Y(t)$ be one-valued continuous functions of t in the interval $\mathfrak{A} = (\alpha < \beta)$. As t ranges over $\mathfrak{A}$ the point x, y will describe a continuous curve C_{ab}. If such a curve has no double points, it will be called a *Jordan curve.* A continuous closed curve without double points will then be a closed Jordan curve. It has been proved that the necessary and sufficient condition for C to be rectifiable, *i. e.*, to have length, is that $X(t)$ and $Y(t)$ have limited variation.†

Let P' and P'' correspond to $t = t'$ and $t = t''$ on the curve C. If $t' < t''$, then we say P' precedes P'' and write $P' < P''$.

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* M. B. Porter, "Concerning Green's Theorem," *Annals of Math.*, Ser. 2, Vol. VII (1905), p. 1.

A. Pringsheim, "Zur Theorie des Doppel-Integrals des Green'schen Integralsatzes," *Sitzungsberichte der k. b. Akademie d. Wissenschaften zu München,* Vol. XXIX (1899), p. 49.

Heffter, "Zur Theorie der reellen Curvenintegrale," *Göttinger Nachrichten* (1902), p. 115.

J. Thomae, "Einleitung in die Theorie der bestimmten Integrale."

W. F. Osgood, "Lehrbuch der Funktionentheorie."

C. Jordan, "Cours d'Analyse," Vol. I.

† Pierpont, "The Theory of Functions of Real Variables," Vol. II, p. 583. Hereafter this will be referred to as Lectures, Vol. II.

As t ranges from α to β, C is described in the positive direction.

If P' on C corresponds to $t=t'$, and at t' either $X(t)$ or $Y(t)$ has a proper extreme,* then the curve C will be said to have a *peak* at P'.

Let C_{ab} be a rectifiable Jordan curve defined by

$$x = X(t), \quad y = Y(t), \qquad t \text{ ranging over } \mathfrak{A} = (\alpha < \beta).$$

A function $f(x, y)$ is defined for each point of C_{ab} and limited over C_{ab}.

Let Δ be a division of $\mathfrak{A}$ of norm δ into subintervals $\delta_1, \delta_2, \ldots, \delta_m$. Then to Δ corresponds a division D of C_{ab} of norm d into arcs $l_1, l_2, \ldots, l_m$. As C is continuous, $d \doteq 0$ with δ.

Let $a = (x^0, y^0), (x^1, y^1), \ldots, (x^m, y^m) = b$ be the end points of arcs $l_1, \ldots, l_m$. Let v_ι be any point on l_ι. Then

$$\int_{C_{ab}} f(x, y)\, dx = \lim_{\delta=0} \Sigma f(v_\iota)(x^\iota - x^{\iota-1}),$$

$$\int_{C_{ab}} f(x, y)\, dy = \lim_{\delta=0} \Sigma f(v_\iota)(y^\iota - y^{\iota-1}),$$

when these limits exist.

Functions for which both limits exist will be called integrable functions over the curve C_{ab}. A sufficient condition for a limited function $f(x, y)$ to be integrable is that $\lim\limits_{\delta=0} \Sigma\, \omega_\iota l_\iota = 0$, where $\omega_\iota = \operatorname{osc} f$ over arc l_ι of C_{ab}.

A set of parallels to the axes will be called a *pantactic* set, if any interval of either the x- or the y-axis is cut by at least one of these parallels.

The field of integration considered in this section will satisfy the following conditions:

1°. The boundary C is a closed rectifiable Jordan curve defined by

$$x = X(t), \quad y = Y(t), \qquad t \text{ in } \mathfrak{A} = (\alpha < \beta).$$

2°. $X(t)$ and $Y(t)$ are functions such that, if A is a discrete set in $\mathfrak{A}$, the image of A given by $X(t)$ or by $Y(t)$ is also discrete.

3°. The points in $\mathfrak{A}$ corresponding to peaks on C form a discrete set.

4°. C has only a finite number of segments parallel to the axes.

5°. C is cut by each one of a pantactic set of parallels in only a finite number of points.

We call such a field a *Green field* and denote it by $\mathfrak{G}$.

We will denote by $\mathfrak{P}_x$ a pantactic set of parallels to the x-axis each of which cuts C only a finite number of times, and does not pass through peaks of C. $\mathfrak{P}_y$ will denote a similar set parallel to the y-axis. We set $\mathfrak{P} = \mathfrak{P}_x + \mathfrak{P}_y$. That such a pantactic set exists, follows from 2°, 3° and 5°, since by 2° and 3° the projections on the axes of the peaks on C are discrete.

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* Lectures, Vol. II, p. 521.

It will be convenient at times to let $\mathfrak{P}_x$, $\mathfrak{P}_y$ denote only the points of $\mathfrak{G}$ on these parallels; in so doing no ambiguity will arise.

A *normal* division D of norm d of a Green field $\mathfrak{G}$ is a division of $\mathfrak{G}$ made by a finite number of parallels of $\mathfrak{P}_x$ and of $\mathfrak{P}_y$. Since $\mathfrak{P}$ is pantactic, d may be made to approach zero.

THEOREM I. *Let C be the boundary of a Green field $\mathfrak{G}$. Let $x=\alpha$, a parallel of $\mathfrak{P}_x$, cut C in the points whose ordinates are $a_1 < a_2 < \ldots . < a_{2n}$. If $x<\alpha$ for points on C immediately preceding $P_\iota=(\alpha, a_\iota)$, then $x>\alpha$ for points on C immediately preceding $P_{\iota+1}=(\alpha, a_{\iota+1})$, and conversely.*

We take $P'=(x', y')$ on C preceding P_ι, so that:

1°. Between P' and P_ι, C does not cut $x=\alpha$.

2°. Dist $(P'P_\iota)<\varepsilon$. (1)

The theorem will be proved for the case where $x'<\alpha$. A similar proof holds where $x'>\alpha$.

Now let $P''=(x'', y'')$ be a point on C so that $P_{\iota+1}<P''$, and between $P_{\iota+1}$ and P'' there is no point of C on $x=\alpha$. Also let dist $(P_{\iota+1}P_\iota)<\varepsilon$. (2)

Suppose that $x<\alpha$ for points immediately preceding $P_{\iota+1}$. Then $x''>\alpha$ for P''.

Let $C_{\iota,\iota+1}$ denote the part of C from a_ι to $a_{\iota+1}$, which does not contain the points P' and P''.

$C_{\iota,\iota+1}$, together with the interval $(a_\iota a_{\iota+1})$, forms a closed Jordan curve Γ, and divides the plane into two precincts,* R and L. (3)

It will be proved later that P' lies in one of these precincts and P'' in the other. For the moment this will be assumed. P' and P'' can be joined by an arc of C, which does not include $C_{\iota,\iota+1}$. Call this arc C'. C' must intersect Γ, as P' is in one precinct and P'' in the other. C has no double points, so C' can not intersect $C_{\iota,\iota+1}$ and must cut the interval $(a_\iota a_{\iota+1})$. Therefore $a_{\iota+1}$ is not the next point to a_ι. This is contrary to the hypothesis. Thus $x>\alpha$ for points immediately preceding $P_{\iota+1}$.

To prove P' and P'' are in different precincts, we take $Q'>P_\iota$ and $Q''<P_{\iota+1}$, so that:

1°. Dist $(Q'P_\iota)$ and dist $(Q''P_{\iota+1})<\varepsilon'$.

2°. C does not intersect $x=\alpha$ between P_ι and Q' or between Q'' and $P_{\iota+1}$. About $P_{\iota+1}$ and P_ι circles ζ_ι and $\zeta_{\iota+1}$, of radius ρ, can be drawn, so small that they will contain no points of $C_{\iota,\iota+1}$ except such as lie between P_ι and Q' or

* Lectures, Vol. II, p. 612.

between Q'' and $P_{\iota+1}$. Within $(a_\iota a_{\iota+1})$ take an interval $\lambda = (a_\iota + \eta, a_{\iota+1} - \eta)$, $\eta < \rho$. Then

$$\mu = \text{dist}\,(\lambda\, C_{\iota,\iota+1}) \text{ is } > 0. \tag{4}$$

Starting with $a_\iota + \eta$ as the first center, draw a series of equal overlapping circles $c_1, c_2, \ldots.$ with centers $m_1, m_2, \ldots.$ on λ, with radius $r < \mu$. Finally, each point of λ will lie in at least one circle. The number of circles required will be finite. Let it be m.

$a_\iota + \eta$ is a frontier point of Γ; therefore c_1 contains points of both R and L. Since c_1 contains no point of $C_{\iota,\iota+1}$ as $r < \mu$, any two points of c_1 for which $x > \alpha$ can be joined by a curve not cutting Γ. Thus all such points belong to one precinct. The same statement is true for points of c_1, where $x < \alpha$. A point in c_1 as M for which $x > \alpha$, and a point in c_1 as N for which $x < \alpha$, can not belong to the same precinct, because then all points of c_1 would belong to the same precinct. Of the two precincts L, R introduced above, let L denote that precinct to which the points in c_1 belong for which $x < \alpha$. (5)

Similar reasoning can be applied to each circle, since the centers are frontier points of Γ. This reasoning proves that in any circle all the points for which $x < \alpha$ belong to one precinct, and all those where $x > \alpha$ to the other. (6)

Some points of c_1 lie in ζ_ι, since $\eta < \rho$. Let us take a set of points $p_0, p_1, \ldots, p_m$ for which $x < \alpha$, and such that p_0 is in ζ_ι and c_1, p_1 in c_1 and c_2, etc. Finally, p_m is in c_m and $\zeta_{\iota+1}$. From (5), p_0 and p_1 belong to L. From (6), p_1 and p_2 belong to the same precinct. Thus p_0, p_1, p_2 are all in L.

Continuing in this way, we can prove that all the p points belong to L. (7)

Take a corresponding set $q_0, q_1, \ldots, q_m$ for which $x > \alpha$. Reasoning on these points as we did on the others, we can prove that they belong to R. (8)

In ζ_ι there is no part of $C_{\iota,\iota+1}$ for which $x < \alpha$; therefore all points in ζ_ι for which $x < \alpha$ belong to one precinct. From (7), p_0 belongs to L; thus all points in ζ_ι for which $x < \alpha$ belong to L.

From (1), P' is in ζ_ι if $\varepsilon < \rho$, and $x' < \alpha$ by hypothesis. Therefore P' belongs to L.

Similarly all points in $\zeta_{\iota+1}$ for which $x > \alpha$ belong to one precinct. From (8), q_m belongs to R; therefore all points in $\zeta_{\iota+1}$ where $x > \alpha$ belong to R.

From (2), P'' is in $\zeta_{\iota+1}$ if $\varepsilon < \rho$, and we assumed that $x'' > \alpha$. Thus P'' belongs to R.

Therefore P' and P'' are in different precincts.

Remark. A similar theorem can be proved for $y = \beta$, any parallel of $\mathfrak{P}_y$.

We can now assume that the positive direction on C is such that, if (α, a_1)

is the first point of C on $x=\alpha$, a parallel of $\mathfrak{P}_x$, then $x<\alpha$ for points on C immediately preceding (α, a_1). For if this were not the case, the variable t could be replaced by a new variable $-t$.

THEOREM II. *Let C be the boundary of a Green field $\mathfrak{G}$. Let $x=\alpha$ and $y=\beta$ be parallels of $\mathfrak{P}$. Let $y=\beta$ cut C in $Q_1, Q_2, \ldots, Q_{2m}$, reckoned from left to right. Then $y>\beta$ for points on C immediately preceding Q_1.*

Let S be a square containing $\mathfrak{G}$, and bounded by $x=A$, $x=B$, $y=E$, $y=F\cdot$ $A<B$, $E<F$. On C, we take points $V=(v_1, v_2)<Q_1$ and $W=(w_1, w_2)>Q_{2m}$, so that:

1°. Dist (VQ_1) and dist $(WQ_{2m})<\varepsilon$. (1)

2°. C does not intersect $y=\beta$ between V and Q_1 or between Q_{2m} and W. Suppose $y<\beta$ for points on C immediately preceding Q_1. Then Theorem I shows that $y>\beta$ for points immediately preceding Q_{2m}. Then

$$w_2<\beta \text{ and } v_2<\beta. \tag{2}$$

Let $C_{1,2m}$ be the part of C from Q_1 to Q_{2m} not including V or W.

We now take $x=\alpha$ so that $x=\alpha$ cuts $y=\beta$ between the points Q_1 and Q_{2m}, and also $v_1<\alpha<w_1$. (3)

Let λ_1 be the segment of $y=\beta$ from Q_1 to $x=A$. Let λ_{2m} be the segment of $y=\beta$ from Q_{2m} to $x=B$. $C_{1,2m}+\lambda_1+\lambda_{2m}$ and the boundary of S where $y<\beta$ make up the boundary of a precinct H.

Let a_r be the point of $C_{1,2m}$ on $x=\alpha$ nearest $y=E$. Let the segment of $x=\alpha$ from a_r to $y=E$ be τ. τ divides H into two precincts. That precinct which includes λ_1 in its frontier we will denote by L, and the other by R. Then V is in L and W in R.

To prove this, let us take P_1 and P_{2m}, two points on $C_{1,2m}$, so that $Q_1<P_1$ and $P_{2m}<Q_{2m}$. Moreover, between Q_1 and P_1, P_{2m} and Q_{2m} there shall be no point of $C_{1,2m}$ on $y=\beta$. Let $C^*_{1,2m}$ denote the part of $C_{1,2m}$ between P_1 and P_{2m}. Then

$$d_1=\text{dist}\,(\lambda_1 C^*_{1,2m}),\quad d_2=\text{dist}\,(\lambda_2 C^*_{1\ 2m})>0.$$

About Q_1 and Q_{2m} draw circles c_1, c_2 with radius $r_1<d_1$, $r_2<d_{2m}$. If $\varepsilon<r$, V lies in c_1 and W in c_2, from (1).

Since c_1 contains no points of $C^*_{1,2m}$, all points of c_1 where $y<\beta$ must belong to L. But $v_2<\beta$, so V lies in L. Similarly we can prove that W lies in R.

V and W can be joined by an arc of C not including $C_{1,2m}$. Call this arc C_{vw}. C_{vw} must intersect the boundary of L. C_{vw} can not cut the boundary of S, or $C_{1,2m}$, or λ_1, or λ_{2m}; so it must intersect τ at some point a_s. C_{vw} may

cut $x=\alpha$ in other points below a_s, but in order to return to V in L, C_{vw} will have to cut $x=\alpha$ an odd number of times. Then Theorem I proves that $x>\alpha$ for points immediately preceding (α, a_1). This is contrary to the definition of the positive direction on the curve C. Thus the theorem follows.

It is now necessary to introduce the idea of limited fluctuation.*

Let P be one of the parallels of $\mathfrak{P}_x$. Let D be a division of P of norm d. Let $\omega_\iota = \operatorname{osc} f$ over the ι-th interval on P. Suppose now

$$\max \Sigma\, \omega_\iota < F, \text{ for any } P \text{ of } \mathfrak{P}_x. \tag{1}$$

Let a be a discrete set on P. Let $\omega_a = \Sigma\, \omega_\iota$ over those intervals of D containing points of a. Let

$$\omega_a < M \bar{a}_\delta \text{ for any line of } \mathfrak{P}_x, \tag{2}$$

where M is independent of P.

When (1) and (2) are satisfied, $f(x, y)$ has limited fluctuation with respect to y on $\mathfrak{P}_x$.

A similar definition holds for limited fluctuation with respect to x on $\mathfrak{P}_y$.

Let the curve C be the boundary of a Green field $\mathfrak{G}$. We effect a division D of norm d of C by taking points

$$C_1, C_2, \ldots . \tag{1}$$

on it which satisfy the following conditions:

If C has no segments parallel to the y-axis, we will take the points (1) so that they lie on $x=\alpha_1$, $x=\alpha_2$, ..., parallels of $\mathfrak{P}_x$, and such that each arc $C_\iota C_{\iota+1}$ has length $< d$.

If C has segments parallel to the y-axis, we may suppose their lengths are all $< d$. For if the length of any such segment is $\geq d$, we may subdivide it. The end points of these parallel segments also form a part of the division D.

Let the parallels $x=\alpha_\iota$ cut C in the points whose ordinates are

$$a_{\iota,1} < a_{\iota,2} < \ldots . < a_{\iota,2n}.$$

We now prove Theorem III.

THEOREM III. *Let $f(x,y)$ be limited and integrable over the boundary C of a Green field, and have limited fluctuation on $\mathfrak{P}_x$ with respect to y. Then*

$$\lim_{d=0} \sum_{\iota j} \left[f(\alpha_\iota, a_{\iota,2j}) - f(\alpha_\iota, a_{\iota,2j-1}) \right] \Delta x_\iota = -\int_C f(x, y)\, dx, \tag{1}$$

where $\Delta x_\iota = \alpha_{\iota+1} - \alpha_\iota$.

Using the notation in the beginning of this paper,

$$\Sigma f(v_\iota)(x^\iota - x^{\iota-1}) \doteq \int_C f(x, y)\, dx. \tag{2}$$

* A similar classification of functions is given in Lectures, Vol. II, p. 634.

For the sake of brevity let us set $\delta_\iota = x^\iota - x^{\iota-1}$. Three classes of points are to be considered.

(A) *Where $a_{\iota,j}$ and $a_{\iota,j+1}$ are consecutive points of D.* Here $\delta_\iota = 0$, so $\Sigma f(v_\iota)\delta_\iota$ is not affected by such terms. However, in general,

$$f(\alpha_\iota, a_{\iota,j}) \neq f(\alpha_\iota, a_{\iota,j+1}).$$

Let $\sum\limits_A = \sum\limits_{\iota j} [f(\alpha_\iota, a_{\iota,2j}) - f(\alpha_\iota, a_{\iota,2j-1})]\,\Delta x_\iota$ for terms belonging to points of class (A). Then

$$|\sum_A| \leq P \sum_A \Delta x_\iota, \tag{3}$$

where $P = \max \Sigma$ osc f over any line of $\mathfrak{P}_x$.

P is finite, as $f(x, y)$ has limited fluctuation. Since the intervals $\sum\limits_A \Delta x_\iota$ contain the projection of the peaks of C, and this projection is a discrete set,

$$\sum_A \Delta x_\iota < \varepsilon/2P, \qquad d < d'.$$

From (3),

$$|\sum_A| < \varepsilon/2, \qquad d < d'. \tag{4}$$

(B) *Points lying on vertical segments of C.* Let m be the number of such segments. In the sum in (2) there are at most $2m$ terms affected by these points. Denote their sum by $\sum\limits_B f(v_\iota)\delta_\iota$. Then

$$|\sum_B f(v_\iota)\delta_\iota| < \varepsilon/4, \qquad d < d''. \tag{5}$$

Let the sum of the corresponding terms in (1) be $\sum\limits_B$. This contains at most $2m$ terms. Therefore

$$|\sum_B| < \varepsilon/4, \qquad d < d'''. \tag{6}$$

(C) *Where two consecutive division points of D lie on two consecutive parallels $x = \alpha_\iota$, $x = \alpha_{\iota+1}$.* The positive direction on C is such that $x < \alpha$ for points on C immediately preceding $(\alpha_\iota, a_{\iota,1})$. Theorem I proves the same is true for $(\alpha_\iota, a_{\iota,2j-1})$, and also proves that $x > \alpha$ for points on C immediately preceding $(\alpha_\iota, a_{\iota,2j})$. Therefore,

$$f(\alpha_\iota, a_{\iota,2j})\,\Delta x_\iota = -f(v_\iota)\,\delta_\iota,$$
$$f(\alpha_\iota, a_{\iota,2j-1})\,\Delta x_\iota = f(v_\iota)\,\delta_\iota.$$

Thus

$$|-\Sigma f(v_\iota)\delta_\iota - \sum_{\iota j}[f(\alpha_\iota, a_{\iota,2j}) - f(\alpha_\iota, a_{\iota,2j-1})]\,\Delta x_\iota| \leq |\sum_A| + |\sum_B| + |\sum_B f(v_\iota)\delta_\iota|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}, \text{ from (4), (5), (6),}$$
$$< \varepsilon, \qquad \delta < \delta_0, \tag{7}$$

where $\delta', \delta'', \delta''' \geq \delta_0$.

From (2), (7) comes (1).

Remark. If x and y are interchanged in the preceding theorem, and if Theorem II is used, (1) becomes

$$\lim_{d=0} \sum_{\iota j} [f(b_{j,2\iota}, \beta_j) - f(b_{j,2\iota-1}, \beta_j)] \Delta y_j = \int_C f(x, y)\, dy.$$

In the Green field $\mathfrak{G}$, we take a discrete set A. Some cells of a normal division D of norm d of $\mathfrak{G}$ will contain points of A. We will call this sum A_d. A_d will in general consist of one or more columns of cells parallel to the y-axis. The boundaries of these columns will be made up of lines parallel to the axes and parts of C. Let B_d denote the parts of these boundaries which belong to C or are parallel to the x-axis. That is, B_d is the boundary of A_d exclusive of the parts parallel to the y-axis.

For the parallel $x = \alpha_\iota$ of D which cuts B_d, denote the points of intersection by $\zeta_{\iota,1} < \zeta_{\iota,2} < \ldots < \zeta_{\iota,2m}$. In the next theorem, Theorem IV, the ζ's will have the meaning defined above.

THEOREM IV. *Let $f(x, y)$ be limited and integrable over the boundary C of a Green field $\mathfrak{G}$, and have limited fluctuation on $\mathfrak{P}_x$ with respect to y. Then*

$$\sum_{\iota j} [f(\alpha_\iota, \zeta_{\iota,2j-1}) - f(\alpha_\iota, \zeta_{\iota,2j})] \Delta x_\iota \doteq 0 \text{ as } d \doteq 0. \tag{1}$$

Let $\Sigma \Delta' x_\iota$ be the sum of the projections on the x-axis of columns of cells belonging to A_d, where $\sum_j |\zeta_{\iota,j} - \zeta_{\iota,j+1}| >$ some positive number L. Then

$$\lim_{d=0} \sum_\iota \Delta' x_\iota = 0 \text{ for any fixed } L. \tag{2}$$

For suppose the contrary; *i. e.*,

$$\sum_\iota \Delta' x_\iota > M > 0 \text{ for some } L.$$

Then area $A_d \geq ML > 0$. This is contrary to the hypothesis that A is discrete.

Let $\mathfrak{G}_x$ be the projection of $\mathfrak{G}$ on the x-axis. Divide the columns of cells of A_d into two classes according as

$$\text{(a)} \quad \sum_j |f(\alpha_\iota, \zeta_{\iota,2j-1}) - f(\alpha_\iota, \zeta_{\iota,2j})| \leq \varepsilon/2\,\mathfrak{G}_x, \qquad d < d'; \tag{3}$$

$$\text{(b)} \quad \sum_j |f(\alpha_\iota, \zeta_{\iota,2j-1}) - f(\alpha_\iota, \zeta_{\iota,2j})| > \varepsilon/2\,\mathfrak{G}_x, \qquad d < d'. \tag{4}$$

Since $f(x, y)$ has limited fluctuation, for columns of (b) $\sum_j |\zeta_{\iota,j} - \zeta_{\iota,j+1}| >$ some number L. For such columns of cells, from (2),

$$\sum_\iota \Delta' x_\iota < \varepsilon/2P, \qquad d < d'', \tag{5}$$

where $P = \max \Sigma \operatorname{osc} f$ for any line of $\mathfrak{P}_x$. P is finite, as $f(x, y)$ has limited

fluctuation. Then the sum in (1) breaks up into two sums $\sum_a$ and $\sum_b$:

$$\Big|\sum_a [f(\alpha_\iota, \zeta_{\iota,2j}) - f(\alpha_\iota, \zeta_{\iota,2j-1})]\Delta x_\iota\Big| < \mathfrak{G}_x \max \sum_j |f(\alpha_\iota, \zeta_{\iota,2j}) - f(\alpha_\iota, \zeta_{\iota,2j-1})|$$
$$< \varepsilon/2, \qquad d < d_0 \text{ from } (3),$$
$$\Big|\sum_b [f(\alpha_\iota, \zeta_{\iota,2j}) - f(\alpha_\iota, \zeta_{\iota,2j-1})]\Delta x_\iota\Big| < P\sum_\iota{}' \Delta x_\iota < \varepsilon/2, \qquad d < d_0 \text{ from } (4),$$

where d', $d'' > d_0$. Therefore

$$\Big|\sum_{\iota j} [f(\alpha_\iota, \zeta_{\iota,2j}) - f(\alpha_\iota, \zeta_{\iota,2j-1})]\Delta x_\iota\Big| < \Big|\sum_a\Big| + \Big|\sum_b\Big| < \varepsilon, \qquad d < d_0.$$

Remark. In the preceding theorem x and y may obviously be interchanged.

THEOREM V. *Let $\mathfrak{B}$ be a pantactic* set in the Green field $\mathfrak{G}$. Let A be a discrete set in $\mathfrak{G} - \mathfrak{B}$. Let D be a normal division of $\mathfrak{G}$ of norm d. Let $d_1, \ldots, d_m$ be cells of $\mathfrak{G}$ containing no points of A. Let $f(x, y)$ be limited and integrable over $\mathfrak{B}$. Then*

$$\lim_{d=0} \Sigma f(v_\iota)\, d_\iota = \int_{\mathfrak{B}} f(x, y)\, d\mathfrak{B}. \tag{1}$$

Let $e_1, \ldots, e_n$ denote the cells of $\mathfrak{G}$ made by D. Let $\delta_1, \ldots, \delta_r$ denote the cells of $\mathfrak{G}$ containing points of A. Then

$$\Sigma e_\iota = \Sigma d_\iota + \Sigma \delta_\iota.$$

By definition of an integral,

$$\Big|\int_{\mathfrak{B}} f(x, y)\, d\mathfrak{B} - \Sigma f(v_\iota)\, e_\iota\Big| < \varepsilon/2, \qquad d < d'. \tag{2}$$

Also

$$|\Sigma f(v_\iota)\, e_\iota - \Sigma f(v_\iota)\, d_\iota| < F\Sigma\delta_\iota, \tag{3}$$

where $|f| < F$ in $\mathfrak{B}$. Since A is discrete,

$$|\Sigma\delta_\iota| < \varepsilon/2F, \qquad d < d''. \tag{4}$$

From (2), (3), (4),

$$\Big|\int_{\mathfrak{B}} f(x, y)\, d\mathfrak{B} - \Sigma f(v_\iota)\, d_\iota\Big| < \varepsilon, \qquad d < d_0,$$

where $d_0 < d'$, and $d_0 < d''$.

THEOREM VI. *Let $\mathfrak{B}$ be pantactic on each parallel of $\mathfrak{P}_x$, and $\mathfrak{C}$ on each parallel of $\mathfrak{P}_y$, in a Green field $\mathfrak{G}$. Let $B = \mathfrak{P}_x - \mathfrak{B}$ and $C = \mathfrak{P}_y - \mathfrak{C}$ be discrete in two-way space.*

Then $\Delta = \mathfrak{D}v(\mathfrak{B}, \mathfrak{C})$ is pantactic in $\mathfrak{G}$.

Let D be a rectangular division of $\mathfrak{G}$. Let d be a cell of D. As $B + C$ is discrete, and as $\mathfrak{B}, \mathfrak{C}$ are pantactic, there exists in d a rectangle δ which contains only points of $\mathfrak{P}_x$ belonging to $\mathfrak{B}$ and points of $\mathfrak{P}_y$ belonging to $\mathfrak{C}$.

* Pantactic is used here in the sense defined in Lectures, Vol. II, p. 325.

Let δ_x be the part of one of the parallels of $\mathfrak{P}_x$ in δ. Every point of δ_x belongs to $\mathfrak{B}$. The points on δ_x through which pass lines of $\mathfrak{P}_y$ are pantactic. Moreover, each point on $\mathfrak{P}_y$ in δ belongs to $\mathfrak{C}$. Therefore, points of Δ which lie on δ_x are pantactic relative to δ_x. Thus δ and also d contain points of Δ. Since d is any cell of $\mathfrak{G}$, Δ is pantactic relative to $\mathfrak{G}$.

THEOREM VII. *Let $f(x,y)$ be limited and integrable over the boundary C of a Green field $\mathfrak{G}$, and have limited fluctuations on $\mathfrak{P}_x$ with respect to y. Let $\mathfrak{B}$ be the set on $\mathfrak{P}_x$ where $f_2 = \dfrac{\partial f}{\partial y}$ exists. Let $\mathfrak{B}$ be pantactic on each parallel of $\mathfrak{P}_x$ and let $B = \mathfrak{P}_x - \mathfrak{B}$ be a discrete set in $\mathfrak{G}$. Let f_2 be limited and integrable over $\mathfrak{B}$. Then*

$$\int_C f(x,y)\,dx = -\int_{\mathfrak{B}} \frac{\partial f}{\partial y}\,d\mathfrak{B}. \tag{1}$$

Let D be a normal division of $\mathfrak{G}$ of norm d. Let $\mathfrak{A}_d$ be the cells of D where f_2 exists on $\mathfrak{P}_x$, *i. e.*, containing no points of B. Let d_j be a cell of $\mathfrak{A}_d$ one of whose sides lies on $x = \alpha_\iota$ and has end points (α_ι, ν), (α_ι, μ). Since f_2 exists for each point of (ν, μ) on $x = \alpha_\iota$ by hypothesis, the law of the mean may be applied, and we have

$$f(\alpha_\iota, \mu) - f(\alpha_\iota, \nu) = f_2(\alpha_\iota, k_j)(\mu - \nu), \qquad \nu \leq k_j \leq \mu. \tag{2}$$

If we do this for one side of each cell of $\mathfrak{A}_d$, taking always the sides parallel to the y-axis, and add all the equations thus obtained, some of the terms on the left will cancel in pairs. For example, if (α_ι, η) is a point within $\mathfrak{A}_d$, but at the corner of a cell d_j, it will be both a ν point and a μ point. Considering d_j we have

$$f(\alpha_\iota, r) - f(\alpha_\iota, \eta) = f_2(\alpha_\iota, k_j)(r - \eta), \qquad \eta \leq k_j \leq r.$$

Considering d_{j-1} we have

$$f(\alpha_\iota, \eta) - f(\alpha_\iota, \lambda) = f_2(\alpha_\iota, k_{j-1})(\eta - \lambda), \qquad \lambda \leq k_{j-1} \leq \eta.$$

When we add these two equations, $f(\alpha_\iota \eta)$ drops out. In the sum obtained by adding equations of type (2), only terms involving points on the boundary of $\mathfrak{A}_d$ will remain on the left. We call these ξ points, and denote those on $x = \alpha_\iota$ by

$$\xi_{\iota,1} < \xi_{\iota,2} < \ldots\ldots < \xi_{\iota,2r}. \tag{3}$$

These points are finite in number, as D is a normal division. Adding equations of type (2) and multiplying by Δx_ι, we get

$$\sum_{\iota j} [f(\alpha_\iota, \xi_{\iota,2j}) - f(\alpha_\iota, \xi_{\iota,2j-1})]\,\Delta x_\iota = \sum_{\iota j} f_2(\alpha_\iota, k_{\iota,j})\,\Delta x_\iota \Delta y_j. \tag{4}$$

Let $x = \alpha_\iota$ cut C in the points $a_{\iota,1} < a_{\iota,2} < \ldots\ldots < a_{\iota,2n}$. (5)

Let B_d be that part of the boundary of $\mathfrak{G} - \mathfrak{A}_d$ which is not parallel to the y-axis.

Let $x=\alpha_\iota$ cut B_d in $\zeta_{\iota,1}<\zeta_{\iota,2}<\ldots\ldots<\zeta_{\iota,2m}$. (6)

The ξ points are either ζ or a points; *i. e.*, either on C or on B_d.

Suppose $\xi_{\iota,k}=a_{\iota,s}$. Then both k and s are odd or both even. Let k be even. Then, from (3), points near $\xi_{\iota,k}$ but $<\xi_{\iota,k}$ on $x=\alpha_\iota$ belong to $\mathfrak{A}_d$; and those near $\xi_{\iota,k}$ but $>\xi_{\iota,k}$ on $x=\alpha_\iota$ do not. As $\xi_{\iota,k}$ is on C, these latter points do not belong to $\mathfrak{G}$, and the former do. Thus, from (5), s must be even. Similarly, s can be proved odd, when k is odd.

Suppose now $\xi_{\iota,k}=\zeta_{\iota,s}$. If k is even, s is odd and conversely. Let k be even. The statement above holds in regard to points belonging to $\mathfrak{A}_d$. Since $\xi_{\iota,k}$ is not on C, points above $\xi_{\iota,k}$ on $x=\alpha_\iota$ must belong to $\mathfrak{G}-\mathfrak{A}_d$. Then, from (6), s is odd.

Thus the left side of (4) may be broken up into two sums, one containing terms involving ζ points and the other containing terms involving a points. We have, therefore,

$$\sum_{\iota j}[f(\alpha_\iota, a_{\iota,2j})-f(\alpha_\iota, a_{\iota,2j-1})]\Delta x_\iota+\sum_{\iota j}[f(\alpha_\iota,\zeta_{\iota,2j-1})-f(\alpha_\iota,\zeta_{\iota,2j})]\Delta x_\iota=\sum_1+\sum_2. \quad (7)$$

If there exists a point θ which is both an a point and a ζ point, it is not on the boundary of $\mathfrak{A}_d$; so the term $f(\alpha_\iota,\theta)$ does not appear in (4).

Let $\theta=a_{\iota,s}=\zeta_{\iota,r}$. Reasoning similar to that used for the case where $\xi_{\iota,k}=a_{\iota,s}$ shows that s and r are both even or both odd. Therefore, in (7) the term involving $f(\alpha_\iota,\theta)$ enters twice, but with opposite signs; so (7) equals the left side of (4), no matter how many points like θ there may be. Therefore,

$$\sum_1+\sum_2=\sum_{\iota j} f_2(\alpha_\iota, k_{\iota,j})\,\Delta x_\iota\,\Delta y_j. \quad (8)$$

(7) contains now all the terms involving a points for lines $x=\alpha_\iota$ of D and all ζ points. By Theorem III,

$$\sum_1 \doteq -\int_C f(x,y)\,dx. \quad (9)$$

By Theorem IV,

$$\sum_2 \doteq 0. \quad (10)$$

By Theorem V,

$$\sum_{\iota j} f_2(\alpha_\iota, k_{\iota,j})\,\Delta x_\iota\,\Delta y_j \doteq \int_{\mathfrak{B}} \frac{\partial f}{\partial y}\,d\mathfrak{B}. \quad (11)$$

The theorem follows from (8), (9), (10), (11).

Remark. If in the preceding theorem and demonstration x and y are interchanged and Theorem II is used, then (1) becomes

$$\int_C g(x,y)\,dy=\int_{\mathfrak{C}} \frac{\partial g}{\partial x}\,d\mathfrak{C}. \quad (12)$$

We have, as a corollary of the foregoing,

THEOREM VIII. *Let* $\Delta = \mathfrak{D}v\,(\mathfrak{B}, \mathfrak{C})$. *Then, from* (1) *and* (12) *and Theorem* VI,

$$\int_C \{ f(x, y)\, dx + g(x, y)\, dy \} = \int_\Delta \left\{ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\} d\Delta. \tag{13}$$

The relation (13) is Green's theorem proved for a very general class of functions over a field which may be cut, by each one of a certain set of parallels, in an infinite number of points. This set may contain an infinite number of lines. The functions do not need to be continuous with respect to one variable on all lines in the field, but only on a certain set of lines $\mathfrak{P}_x$ or $\mathfrak{P}_y$. The value of the function on lines not belonging to this set is of no consequence, provided the function is integrable over the boundary of the field. The derivative need not exist for every point on the lines $\mathfrak{P}_x$ or $\mathfrak{P}_y$, but must exist for a certain set, over which it is integrable. The derivative may exist for points in the field not on $\mathfrak{P}_x$ or $\mathfrak{P}_y$. When this is true, the derivative may not be integrable over the whole field for which it exists.

§ 2. *Nonrectifiable Green Fields.*

Up to the present, line integrals have been defined only for rectifiable curves. If now we look at the relation (12) obtained in Theorem VIII, we see the right side has a meaning whether the boundary of $\mathfrak{G}$ is rectifiable or not. Let us see, then, if it is not possible to extend our definition of a line integral so that the left side of this relation has a meaning when the curve C is not rectifiable. This can be done for a class of curves defined as follows:

1°. C_{ab} is a Jordan curve defined by

$$x = X(t), \quad y = Y(t), \qquad t \text{ in } \mathfrak{A} = (\alpha < \beta).$$

2°. A is a discrete set in $\mathfrak{A}$ having I as its image on C_{ab}.

3°. C_{ab} is rectifiable except in the vicinity of points of I.

By 3° is meant the following: Let Δ be a division of $\mathfrak{A}$ of norm δ. As A is discrete, if $\delta < \delta_0$ there are intervals containing no points of A. Let their sum be $\mathfrak{A}_d$, and let C_δ be the part of C_{ab} corresponding to $\mathfrak{A}_\delta$. Then C_δ is rectifiable for each $\delta < \delta_0$, but C_{ab} is not rectifiable.

An example of such a curve is

$$\begin{aligned} x = t, \quad y &= t \sin \frac{1}{t}, \qquad 0 < t \leq 1, \\ &= 0, \qquad\qquad t = 0. \end{aligned}$$

This curve is rectifiable except for the point $t = 0$.

The set I will be called the singular points of the nonrectifiable curve C_{ab}.

Let $f(x, y)$ be defined and limited over C_{ab}. If $f(x, y)$ is integrable over C_δ for each $\delta < \delta_0$, $f(x, y)$ is said to be regular over C_{ab}.

Let $f(x, y)$ be regular over the nonrectifiable curve C. If

$$\lim_{\delta=0} \int_{C_\delta} f(x, y)\, dx \text{ and } \lim_{\delta=0} \int_{C_\delta} f(x, y)\, dy$$

exist, these limits are denoted by

$$\int_{C_{ab}} f(x, y)\, dx \quad \text{and} \quad \int_{C_{ab}} f(x, y)\, dy.$$

If both limits exist, $f(x, y)$ is said to be integrable over C_{ab}.

A field $\mathfrak{G}$ whose boundary C satisfies the following conditions will be called a *nonrectifiable Green field.*

1°. C is a closed Jordan curve defined by

$$x = X(t), \quad y = Y(t), \qquad t \text{ in } \mathfrak{A} = (\alpha < \beta),$$

and rectifiable except for a set I whose images on the axes are discrete.

2°. For any C_δ the corresponding functions $X_\delta(t)$ and $Y_\delta(t)$ are such that, if A is a discrete set in $\mathfrak{A}_\delta$, the image of A given by $X_\delta(t)$ or by $Y_\delta(t)$ is discrete.

3°. The points at which either $X(t)$ or $Y(t)$ has a proper extreme form a discrete set in $\mathfrak{A}$.

4°. C has only a finite number of segments parallel to the axes.

5°. C is cut by each one of a pantactic set of parallels in only a finite number of points.

Let $f(x,y)$ be regular over the nonrectifiable boundary C of a Green field $\mathfrak{G}$, and have limited fluctuation on $\mathfrak{P}_x$ with respect to y. Then $f(x,y)$ will be called a normal function of $\mathfrak{G}$ with respect to y. A similar definition holds for a normal function with respect to x.

THEOREM IX. *Let $f(x,y)$ be a normal function of the nonrectifiable Green field $\mathfrak{G}$ with respect to y. Let $\mathfrak{B}$ be the set on $\mathfrak{P}_x$ where $f_2 = \dfrac{\partial f}{\partial y}$ exists. Let $B = \mathfrak{P}_x - \mathfrak{B}$ be discrete. Let f_2 be limited and integrable over $\mathfrak{B}$. Then*

$$\int_C f(x, y)\, dx = -\int_{\mathfrak{B}} \frac{\partial f}{\partial y}\, d\mathfrak{B}. \tag{1}$$

Let I_x be the projection of the singular points of C on the x-axis. I_x is discrete by the definition of a nonrectifiable Green field. Let D be a normal division of $\mathfrak{G}$ of norm d. Let us look at the columns of cells parallel to the y-axis, which contain no points of I. In each of these columns, there is one or more partial fields of $\mathfrak{G}$. These are finite in number, as D is a normal division.

Let these fields be $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_m$, with boundaries $\lambda_1, \lambda_2, \ldots, \lambda_m$. Let $\mathfrak{B}_\iota$ be the part of $\mathfrak{B}$ in $\mathfrak{G}_\iota$.

Each $\mathfrak{G}_\iota$ satisfies the conditions of Theorem VII; therefore

$$\int_{\lambda_\iota} f(x, y)\, dx = \int_{\mathfrak{B}_\iota} \frac{\partial f}{\partial y}\, d\mathfrak{B}_\iota. \tag{2}$$

Let l_ι be the part of λ_ι belonging to C. For the rest of λ_ι, $dx = 0$, since it is parallel to the y-axis. Thus

$$\int_{l_\iota} f(x, y)\, dx = \int_{\lambda_\iota} f(x, y)\, dx. \tag{3}$$

Let $\mathfrak{B}_0 = \sum_{\iota=1}^{m} \mathfrak{B}_\iota$, $C_\delta = \sum_{\iota=1}^{m} l_\iota$. From (2), (3),

$$\int_{C_\delta} f(x, y)\, dx = -\int_{\mathfrak{B}_0} \frac{\partial f}{\partial y}\, d\mathfrak{B}_0. \tag{4}$$

As $\overline{\mathfrak{B}}_0 \doteq \overline{\mathfrak{B}}$,

$$\int_{\mathfrak{B}_0} \frac{\partial f}{\partial y}\, d\mathfrak{B}_0 \doteq \int_{\mathfrak{B}} \frac{\partial f}{\partial y}\, d\mathfrak{B}.^* \tag{5}$$

Therefore $\lim\limits_{\delta=0} \int_{C_\delta} f(x, y)\, dx$ exists. Then, by definition,

$$\lim_{\delta=0} \int_{C_\delta} f(x, y)\, dx = \int_C f(x, y)\, dx. \tag{6}$$

Therefore $\int_C f(x, y)\, dx = -\int_{\mathfrak{B}} \frac{\partial f}{\partial y}\, d\mathfrak{B}$, from (4), (5), (6).

Remark. If x and y are interchanged as has been done before, (1) becomes

$$\int_C g(x, y)\, dy = \int_{\mathfrak{C}} \frac{\partial g}{\partial x}\, d\mathfrak{C}. \tag{7}$$

As a corollary of the preceding, we have

THEOREM X. *Let* $\Delta = \mathfrak{D}v(\mathfrak{B}, \mathfrak{C})$. *Then*

$$\int_C \{f(x, y)\, dx + g(x, y)\, dy\} = \int_\Delta \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] d\Delta. \tag{8}$$

(8) follows at once from (1) and (7).

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* Lectures, Vol. II, pp. 55, 60.